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The BPS spectrum of $\mathcal{N} = 2$ SU(N) SYM

Bartomeu Fiol

New High Energy Theory Center, Rutgers University
Piscataway, NJ 08855, U.S.A.
E-mail: fiol@physics.rutgers.edu

Abstract: We apply ideas that have appeared in the study of D-branes on Calabi-Yau compactifications to the derivation of the BPS spectrum of field theories. In particular, we identify an orbifold point whose fractional branes can be thought of as "partons" of the BPS spectrum of $\mathcal{N} = 2$ pure SU(N) SYM. We derive the BPS spectrum and lines of marginal stability branes near that orbifold, and compare our results with the spectrum of the field theories.

Keywords: Supersymmetric gauge theory, Duality in Gauge Field Theories.

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1. Introduction

Over the last year, a framework for the determination of the classical spectrum of BPS branes of IIA string theory on Calabi-Yau varieties has been developed, valid throughout the compactification moduli space (see [1] for the state of the art of this program and an extensive list of references). One of the main ingredients of the emerging picture is that we can think of the BPS spectrum as boundstates of a finite number of "parton" branes (e.g., fractional branes near an orbifold, $L=0$ boundary states at a Gepner point). These parton branes are rigid, in the sense that they have no moduli space. Mathematically, they provide a basis for the K theory of the Calabi-Yau. There has been much work devoted to a better understanding of this finite set of branes for different compactifications, and the determination of their large volume charges [2–9].

Among the many applications of D-branes on Calabi-Yau manifolds, a specially fruitful one has been the derivation of many non-trivial nonperturbative results of $\mathcal{N}=2$ quantum field theories [10, 11], building on earlier work [12]. By suitably choosing a local compactification geometry and taking a decoupling limit, one can study a host of field theories with different gauge groups and matter content. This philosophy has come to be known as "geometric engineering of quantum field theories" [13], and it has a number of advantages: it is very systematic, gives a rationale for the unexpected appearance of geometrical objects in the Seiberg-Witten [14] solution of these theories, and allows the study of new theories without a known Lagrangian formulation.

In the present paper we relate recent developments in the study of D-branes on Calabi-Yau manifolds with the BPS spectrum of $\mathcal{N} = 2$ field theories. The essential idea is the following; for concreteness, we will consider the case of pure SU(N) SYM. Recall that along the moduli space of $\mathcal{N} = 2$ SU(N) SYM, there is a set of $N(N-1)$ monopoles or dyons that can go massless, and we can pick up $2(N-1)$ of them to form a basis of vanishing cycles [15]. We claim that these $2(N-1)$ potentially massless dyons constitute another example of "partons", and the rest of the spectrum can be thought of as boundstates of them. For instance, for SU(2), the monopole and the fundamental dyon that go massless in the strong coupling constitute such a set of "partons", and the W^+ and the tower of dyons present in the weak coupling appear as boundstates of monopoles and fundamental dyons. From the string theory point of view, this amounts to a shift in perspective with respect to geometric engineering: the starting point of geometric engineering is given by a IIA compactification on a geometry of 2-cycles, and D2-branes wrapping about them, that correspond to the perturbative (electric) degrees of freedom of the field theory. Here, we compute the BPS spectrum in the non-geometric phase of the string theory compactification; in particular, we identify the parton branes, which correspond to D4-branes wrapping 4-cycles, with the basis of vanishing cycles in the field theory, so we obtain a description of the field theory spectrum in terms of magnetic degrees of freedom.[1]

Our strategy is the following: we start by identifying an orbifold point $\mathbb{C}^3/\mathbb{Z}_{2N}$ in the moduli space of the non-compact Calabi-Yau used to geometrically engineer SU(N). Once we have the worldvolume theory of the branes at that orbifold, we can in principle determine the BPS spectrum of that compactification. As explained in [16, 4, 17] near the orbifold this is a two step process: if we have a set of k different kinds of "partonic" branes, first we have to determine for which values of $(n_1, \ldots, n_k)$ there is a vacuum configuration, compatible with the superpotential, breaking the original gauge group $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k)$ down to U(1). If there is such a configuration for $(n_1, \ldots n_k)$, then a boundstate of n_1 times the first parton, n_2 times the second parton and so on, can exist somewhere in moduli space. The second part of the procedure is to find where in moduli this state exists. The answer depends on the Fayet-Iliopoulos terms and goes by the name of θ-stability [18, 16].

As we will show, in a particular neighborhood of the orbifold, we are able to identify the spectrum of BPS branes with the BPS states of the SU(N) SYM field theory. The strong coupling spectrum of these field theories was recently derived by Lerche [19], by considering boundary states of a Gepner model in the mirror Calabi-Yau.

It is clear from particular examples [4], that once we move sufficiently away from the orbifold, it is generically not true that the BPS spectrum can be described as boundstates of positive numbers of fractional branes. The reason is the following: each fractional brane

[1] This is always on the type IIA side. In the mirror type IIB side, the electric magnetic duality of the theory is manifest, as D3 branes correspond to both electric and magnetic particles in the field theory.

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has a central charge whose phase determines which particular $\mathcal{N} = 1$ supersymmetry is preserved, from the bulk $\mathcal{N} = 2$. The crucial claim is that boundstates of different fractional branes can be described by a (softly broken) $\mathcal{N} = 1$ theory, even though each fractional brane in the boundstate may have different phase for the central charge. At the orbifold point, all the central charges of the fractional branes are parallel, so in the neighborhood of the orbifold the differences among the phases is small and the previous claim is justified. As we move away from the orbifold, eventually the phases of the central charges differ significantly, and we can encounter boundstates of branes that at the orbifold had antiparallel central charges.[2] In the light of these remarks, it is not *a priori* obvious that an analysis near the orbifold should suffice to recover the field theory spectrum. A better understanding of why this is the case would require considering the periods of these Calabi-Yau backgrounds.

The organization of the paper is the following. In section 2, after briefly recalling the Calabi-Yau used for the geometric engineering of SU(N), we describe an orbifold point in the moduli space of that Calabi-Yau, and derive the worldvolume theory of the D-branes at that orbifold. Our next task is to obtain the spectrum of classical boundstates arising from that worldvolume theory, and discuss the jumps in the spectrum that take place near the orbifold; that we do in section 3, and in section 4 we compare our results with the spectrum of SU(N) and the known lines of marginal stability. In section 5, we state our conclusions.

2. $\mathcal{N} = 2$ SU(N) SYM and fractional branes

In this section we identify a non-compact orbifold, $\mathbb{C}^3/\mathbb{Z}_{2N}$, as the non-geometric phase of the local IIA Calabi-Yau compactification used for the geometric engineering of $\mathcal{N} = 2$ SU(N) SYM [13]. Once we have identified that orbifold, we can use the techniques of [20] to derive the $\mathcal{N} = 1$ worldvolume theory for branes on that geometry. In subsequent sections, we will analyze the spectrum of boundstates of those theories.

2.1 Geometric engineering of $SU(N)$

It will be useful to recall the setup for the geometric engineering of SU(N).[3] The starting point is to consider an A_{N-1} singularity in six dimensions, since at this singularity, type IIA develops an enhanced SU(N) gauge symmetry. Next, we want to compactify down to four dimensions, breaking half of the supersymmetry on the way. To accomplish this, we fiber the A_{N-1} singularity over a base $\mathbb{P}^1$. Recall that a A_{N-1} singularity can be thought of as a $\mathbb{C}^2/\mathbb{Z}_N$ orbifold blown up by $N-1$ $\mathbb{P}^1$'s. We will denote this geometry of $N-1$ $\mathbb{P}^1$'s fibered over a base $\mathbb{P}^1$ by X.

Since we are interested in extracting field theory results from this compactification of string theory, we need to decouple the effects of gravitational and massive string excitations.

[2] The natural way of keeping track of this possibility is by considering the derived category of the category of quiver representations [1]. This introduces a (useful!) redundancy in the description, and in this sense resembles a gauge symmetry.

[3] For reviews on geometric engineering, see [21, 15, 22].

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This amounts to taking the limit where the size of the base $\mathbb{P}^1$ goes to ∞, whereas the sizes of the $\mathbb{P}^1$'s in the fiber go to zero. This keeps finite the mass of the $W^\pm$ bosons, which arise from D2-branes wrapping the 2-cycles of the fiber $\mathbb{P}^1$'s. On the other hand, the magnetically charged states in the field theory arise from branes wrapping 4-cycles, and in the limit that the volume of the base is sent to infinity, these states decouple from the perturbative theory.

Ultimately, we are interested in the vector moduli space of this compactification, and for this purpose it is crucial to consider type IIB on the mirror geometry $\hat{X}$. The reason for this is that, while for both type IIA on X and type IIB on $\hat{X}$, the vector moduli space is free of quantum string corrections (as in both cases the dilaton sits in a hypermultiplet), on the type IIA side we would have to deal with world sheet instanton corrections, whereas on the type IIB side there are no such corrections. An easy way to understand this is to note that the vector moduli space for type IIB encodes the size of the 3-cycles of $\hat{X}$, and neither the fundamental strings nor the IIB BPS D-branes can wrap a 3-cycle to produce an instanton. Therefore, a purely classical description of type IIB on $\hat{X}$ encodes all the nonperturbative physics of the quantum field theory.

It is convenient to describe this geometry by a two dimensional linear sigma model [23]. This description, or more precisely, the equivalent toric diagram, will be specially helpful to determine the non-geometric phase. As we just reviewed, for SU(N) with no matter, we have to consider IIA on a A_{N-1} ALE singularity fibered over a $\mathbb{P}^1$. For each 2-cycle we introduce a U(1), and since we have a $\mathbb{P}^1$ in the base and $N-1$ $\mathbb{P}^1$'s as fibers, the gauge group will be $\mathrm{U}(1)^N$. The matter content is given by $N+3$ chiral fields, whose charge vectors with respect to the N U(1)'s are

$$\begin{aligned} v_b &= (1,1,-2,0,0,\dots,0) \\ v_{f_1} &= (0,0,1,-2,1,\dots,0) \\ &\vdots \\ v_{f_{N-1}} &= (0,0,0,\dots,1,-2,1) \end{aligned}$$

The geometry under consideration does not have odd cycles, and this translates into not having a superpotential in the linear sigma model. The toric diagram has vertices given by

$$\nu_i = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ N & 0 \end{pmatrix}$$

This is displayed in figure 1 for SU(3). The $N-1$ interior points correspond to the $N-1$ compact divisors.

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2.2 The orbifold limit

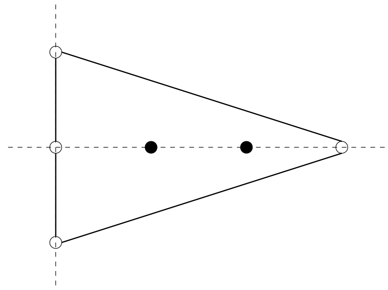

Figure 1: The toric diagram for SU(3).

So far, we have described the standard way to extract $\mathcal{N} = 2$ SU(N) SYM from a Calabi-Yau. An important ingredient is that the charged spectrum of the field theory appears from considering branes wrapping cycles of a Calabi-Yau, a topic which has received a lot of attention lately.[4] One of the central ideas in the emerging framework is that the whole spectrum of BPS states can be thought of as boundstates of a finite set of branes. We would like to determine this set of "parton" branes for the geometry just described, and relate it with the spectrum of SU(N) $\mathcal{N} = 2$ SYM. To do so, we need to go to a point in the moduli space of this geometry where we have a handle on the spectrum of BPS D-branes and their worldvolume theories. To accomplish this, we take a different limit that the one just described: keeping the $\mathbb{P}^1$'s in the fiber blown down, we shrink the size of the base $\mathbb{P}^1$ of X. Formally, in the $t_b \to -\infty$, we reach a solvable point. The resulting non-geometic phase can be described as follows [26]:[5] take the vertices of the toric diagram in $\mathbb{Z}^3$, $(0,-1,1),(0,1,1),(N,0,1)$, and equate the corresponding monomials to $(1,1,1)$

$$(t_3^N, t_1^{-1}t_2, t_1t_2t_3) = (1,1,1)$$

the solution $t_1 = t_2 = \epsilon$, $t_3 = \epsilon^{-2}$, with $\epsilon^{2N} = 1$ describes the orbifold $\mathbb{C}^3/\mathbb{Z}_{2N}$ with spacetime action given by

$$(z_1, z_2, z_3) \longrightarrow (e^{\frac{2\pi i}{2N}} z_1, e^{\frac{2\pi i}{2N}} z_2, e^{-2\frac{2\pi i}{2N}} z_3)$$

This orbifold is the point in the moduli space of the geometry X we are going to focus on.[6] There are $2N$ fractional branes at this orbifold point, and they constitute what we call the basis of parton branes for the BPS spectrum for this compactification. To better understand the relation of this orbifold with the geometry we started with, an A_{N-1} fibration over $\mathbb{P}^1$, we can study the homology of this orbifold. If we denote the generator of $\mathbb{Z}_{2N}$ by g, the action of the kth element of $\mathbb{Z}_{2N}$ is

$$g^k \ : \ (z_1, z_2, z_3) \longrightarrow (e^{k\frac{2\pi i}{2N}} z_1, e^{k\frac{2\pi i}{2N}} z_2, e^{-2k\frac{2\pi i}{2N}} z_3)$$

There is a complex line of $\mathbb{C}^2/\mathbb{Z}_2$ singularities, $(0,0,z_3)$, caused by g^N. We can compute the orbifold cohomology following [27], and the twisted sector contribution is $h^{1,1} = N$, $h^{2,2} = N$. However, one of the elements of $h^{1,1}$, the one coming from the g^N twisted sector, is not a normalizable form on the resolved space, so as in [3], we conclude that it does not correspond to a compact 4-cycle. All told, we have N 2-cycles and $N-1$ compact 4-cycles,

[4] see [24] and [25] for overviews on the physics and the mathematics involved, respectively.

[5] We arrived to this result by different arguments than those presented here. I am indebted to S. Katz for explaining this method to me, and for providing me with the lecture notes [26] prior to publication.

[6] The reason we obtained an orbifold is that the toric diagram we started with was simplicial [26].

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which indeed matches the homology of A_{N-1} fibered over $\mathbb{P}^1$. The picture is then that for each point in $(0,0,z_3)$ we have a $\mathbb{P}^1$, forming a non-compact 4-cycle $\mathbb{C}\times\mathbb{P}^1$, but at the origin $(0,0,0)$ there are extra shrunk cycles.

What is the relation between this orbifold and the field theory?[7] The vector moduli space of this string compactification is N complex dimensional, whereas the moduli space of the corresponding field theory is $N-1$ complex dimensional, and can be regarded as a hypersurface in the former one. In particular, the orbifold point we just described is not sitting in the moduli space of the field theory, and one might worry that, starting at the orbifold, by the time we get to the hypersurface that corresponds to the field theory, the phases of the central charges have changed enough as to render the quiver theory approximation invalid. We should then consider the flow of the gradings and the derived category [1] to study the spectrum. A better understanding of why a particular neighborhood of the orbifold reproduces the expected spectrum of the field theory would require a full analysis of the moduli space and periods of this string theory compactification.

Another point to take into account is that as we shrink the base $\mathbb{P}^1$, we open the possibility for D2 branes to wrap that 2-cycle, yielding new $W^{\pm}$ not present in the weak coupling. The appearance of this nonperturbative $\mathrm{SU}(2)_{\mathrm{base}}$ in the strong coupling limit of geometric engineering of $\mathrm{SU}(N)$ was discussed in [11, 21], building on earlier work [28, 29]. When we compare the spectrum of this string theory compactification with that of the $\mathrm{SU}(N)$ field theory, we have to identify which fractional branes wrap the 2-cycle corresponding to the shrunk $\mathbb{P}^1_{\mathrm{base}}$ and discard them from our discussion.

2.3 The worldvolume theory

Finally, we are ready to derive the $\mathcal{N}=1$ theory of the branes at the orbifold. To do so, we apply the techniques introduced in [20]. Recall that in an orbifold $\mathbb{C}^n/\Gamma$, there is a 1-to-1 correspondence between fractional branes and irreducible representations r_i of Γ. In the present case we have $|\Gamma|=2N$ different fractional branes; if we want to consider a configuration with n_1 fractional branes of the first kind, n_2 fractional branes of the second and so on, we need to take as representation $R=\sum_i n_i\, r_i$. The result is a $\mathcal{N}=1$ theory, with gauge group $\mathrm{U}(n_1)\times\cdots\times\mathrm{U}(n_{2N})$, $2N$ chiral fields $X_{i,i+1}$ transforming in $(n_i,\bar{n}_{i+1})$, $2N$ chiral fields $Y_{i,i+1}$ transforming in $(n_i,\bar{n}_{i+1})$, and $2N$ chiral fields $Z_{i,i-2}$ transforming in $(n_i,\bar{n}_{i-2})$.

This is represented in figure 2 for the $N=3$ orbifold, $\mathbb{C}^3/\mathbb{Z}_6$. The superpotential of the gauge theory is given by the usual reduction of the original $\mathcal{N}=4$ one,

$$\mathcal{W}=\mathrm{tr}\,\sum_{i=1}^{2N}(X_{i,i+1}Y_{i+1,i+2}-Y_{i,i+1}X_{i+1,i+2})Z_{i+2,i}$$

which leads to the F-flatness conditions

$$\begin{aligned}X_{i,i+1}Y_{i+1,i+2}&=Y_{i,i+1}X_{i+1,i+2}\\ Z_{i+2,i}X_{i,i+1}&=X_{i+2,i+3}Z_{i+3,i+1}\\ Z_{i+2,i}Y_{i,i+1}&=Y_{i+2,i+3}Z_{i+3,i+1}\end{aligned}\qquad(2.1)$$

[7] I would like to thank D.E. Diaconescu and C. Vafa for discussions on this point.

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In addition to the superpotential, the $\mathcal{N}=1$ worldvolume theory admits Fayet-Iliopoulos terms for the $2N$ U(1) factors, ζ_i, $i=1,\ldots,2N$. These FI terms can be written in terms of the NS twist fields via a discrete Fourier transform [30]. In the previous subsection we computed the orbifold cohomology, or put differently, the RR ground states. The NS twist fields ϕ_k are related by supersymmetry, and in principle we have 2N complex NS fields, 2 coming from the g^N twisted sector, and one from each of the remaining 2N-2 twisted sectors. There is a reality condition, $\phi_k = \phi^*_{2N-k}$, so finally we have N complex NS fields. These correspond to the N complexified Kähler moduli. Now the FI terms can be read from the coupling [30]

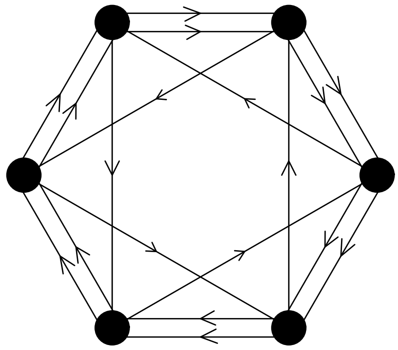

Figure 2: The quiver diagram for the $N=3$ orbifold.

$$\sum_k \int \phi_k \mathrm{Tr}\ \gamma(g^k) D$$

where D is the matrix of auxiliary fields. The outcome is that for N odd, we have two relations $\sum_{k\, odd} \zeta_k = \sum_{k\, even} \zeta_k = 0$ whereas for N even, we only have one relation $\sum_k \zeta_k = 0$. This comes about because for N odd $\mathbb{Z}_{2N} = \mathbb{Z}_2 \times \mathbb{Z}_N$, but the same is not true for even N. The upshot is that we have 2N-2 FI independent terms for N odd and $2N-1$ for N even. In any case, we conclude that the D-branes can not explore the whole of the Kähler moduli space, which is N complex dimensional.

What is the relation between the spectrum of this string compactification and the spectrum of $\mathcal{N}=2$ SU(N)? To identify the fractional branes with states in the field theory our guide will be the intersection matrix, $I_{a,b} = \mathrm{Tr}_{ab}\,(-1)^F$ [31, 2], since in the four dimensional field theory, when the D-branes reduce to particles, $I_{a,b}$ corresponds to the Dirac-Schwinger-Zwazinger (DSZ) product.

Now, since the arrows in the quiver stand for (the bosonic partners) of fermionic massless zero modes, one can suspect that it is possible to read off the intersection matrix of the fractional branes of an orbifold from the quiver. Indeed, for an abelian orbifold $\mathbb{C}^n/\Gamma$ with spacetime action $z_i \to e^{\frac{2\pi i}{|\Gamma|} w_i} z_i$, if we denote by g the $|\Gamma| \times |\Gamma|$ shift matrix, the intersection matrix for the fractional branes is

$$\mathrm{I}_{ab} = \Pi\ (1 - g^{w_i})$$

which for a Calabi-Yau n-fold ($\sum w_i = 0$ mod n), is completely symmetric or antisymmetric depending on the parity of n. In our case,

$$I_{a,b} = (1-g)(1-g)(1-g^{-2}) = -2g + 2g^{-1} - g^{-2} + g^2 \tag{2.2}$$

What makes this intersection matrix relevant for our discussion is that it is exactly (minus) the intersection matrix of vanishing cycles of SU(N), or put differently, the DSZ product for a basis of the potentially massless dyons of SU(N), with magnetic and electric

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charges [19],

$$\begin{pmatrix} [\alpha_1, 0] \\ [-\alpha_1, \alpha_1] \\ \vdots \\ [\alpha_i, (i-1)\alpha_i] \\ [-\alpha_i, (2-i)\alpha_i] \\ \vdots \\ [\alpha_N, \sum(1-k)\alpha_k] \\ [-\alpha_N, \sum(k-2)\alpha_k] \end{pmatrix} \tag{2.3}$$

where α_i, $i = 1, \dots, N-1$ are the simple roots of $su(N)$ and $\alpha_N = -\sum_i \alpha_i$. This suggests that the fractional branes we found at the orbifold correspond, in the field theory limit, to dyons whose magnetic charges are simple roots of the $su(N)$ algebra, and whose electric charges can be chosen as in (2.3). It was stablished in [32], that, at least in the weak coupling, all the particles of SU(N) SYM have magnetic charge a root of the $su(N)$ algebra, and it seems quite natural that those whose magnetic charge is a simple root can play the role of partons for the rest.

Note that for $su(N)$ we have $N-1$ positive simple roots, and indeed the Seiberg-Witten solution for SU(N) is given in terms of a $g = N-1$ Riemann surface with $2(N-1)$ independent 1-cycles [15]. To recover the states with negative magnetic charge, with respect to these $2(N-1)$, we can choose to add two extra 1-cycles with α_N, as in (2.3), or stick just to a set of independent cycles and allow for negative coefficients. This last option is more in the line of [4], where antiparticles near the orbifold came from quivers representations with all the n's negative. This leads us to identify $2(N-1)$ of the $2N$ fractional branes at the orbifold, with the independent vanishing cycles and the corresponding field theory particles. Furthermore, the quiver formed by the two adjacent nodes that we take away gives, by the Beilinson construction [33] , the coherent sheaves over $\mathbb{P}^1$. Since the stable sheaves on $\mathbb{P}^1$ can be identified with the BPS spectrum of SU(2), it is reasonable to assume that these two 2 fractional branes are charged under the base $\mathbb{P}^1$, and we should discard them in discussing the relation of the BPS D-brane spectrum with the SU(N) spectrum. The upshot of this discussion is that we truncate the quiver gauge theory, cutting out two adjacent nodes, and keeping $2N-2$ nodes.

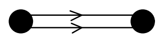

Figure 3: The quiver diagram for the N=3 orbifold.

The intersection matrix (2.2) has also appeared recently [19] in the study of a Gepner point in the moduli space of the type IIB mirror geometry, $\hat{X}$, to which we turn our attention next.

2.4 The mirror picture

Recently, Lerche [19] considered a Gepner point in the moduli space of the type IIB geometry $\hat{X}$. It is claimed in [19] that that Gepner point corresponds to the origin of SU(N) moduli space. This moduli space is $N-1$ complex dimensional, and at the origin there is a $\mathbb{Z}_{2N}$ global symmetry, $u_k \to e^{\frac{2\pi i}{2N}} u_k$, where u_k, $k = 2, \dots, N$ are the Weyl coordinates [34]. The boundary states of that coset model are then identified with the BPS spectrum of

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$\mathcal{N}=2$ SU(N) SYM at strong coupling. The role of parton branes, played on the IIA side by the fractional branes, is played here by the $L=0$ A-type rational boundary states. The starting point of [19] is a LG potential

$$W = x^N + \frac{1}{z_1^{2N}} + \frac{1}{z_2^{2N}} - \sum_{k=2}^{N} u_k x^{N-k} (z_1 z_2)^{-k}$$

where u_k are coordinates for the $N-1$ complex dimensional moduli space. In particular, the point $u_k = 0$ corresponds to the coset model

$$\left(\frac{\mathrm{SU}(2)_{N+2}}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2)_{2N+2}}{\mathrm{U}(1)} \times \frac{\mathrm{SL}(2)_{2N+2}}{\mathrm{U}(1)} \right) / \mathbb{Z}_{2N}$$

The main point of [19] was to prove that the A-type $L=0$ rational boundary states of this Gepner model have the same intersection matrix (2.2) than the basis (2.3) of vanishing cycles of $\mathcal{N}=2$ SU(N) SYM. This follows if we grant that the factors in the intersection matrix coming from the different minimal models, all diagonalize in the same basis, something that for the ordinary minimal models happens for the B-type boundary states [2]. It would be interesting to derive this rule from a careful analysis of this coset model. Note also that the mirror geometry $\hat{X}$ does not have any even cycles. This should translate into the fact that this coset model does not have any B-type boundary states.

3. The spectrum of BPS states near the orbifold

In the previous section, we derived the worldvolume theory for a configuration of $(n_1, \ldots, n_{2N})$ fractional branes. The next question we would like to ask is in which cases they form a BPS boundstate, and how the answer may change as we move in moduli space. The general procedure was introduced in [16, 4], and explained in detail in [17], so here we will be quite brief.

First, the criterion for having a boundstate is that the vevs of the chiral fields break the original gauge group completely, except for the diagonal U(1), which is always present for these theories, and will represent the center of mass motion.

The next thing that we require to the boundstates is that they are BPS. Away from the orbifold, the general configuration of different fractional branes will break all supersymmetry, as each preserves a different $\mathcal{N}=1$ subalgebra of the original $\mathcal{N}=2$. The claim is that this supersymmetry breaking is quite a mild one, caused by a constant non-zero potential coming entirely from D terms. In the language of θ-stability [18, 16] this means that for a given set of values $(n_1, \ldots, n_k)$, we look for θ-stable configurations with the components of $\vec{\theta}$ related to the physical FI terms ζ by

$$\theta_i = \zeta_i - \frac{\vec{n} \cdot \vec{\zeta}}{\vec{n} \cdot e} \tag{3.1}$$

where $e = (1, \ldots, 1)$. The difference between ζ_i and θ_i gives precisely the constant shift in the potential just discussed.

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The last ingredient in the picture is how the spectrum changes as we move in Kähler moduli space. The field theory counterpart would be the determination of the lines of marginal stability in the quantum moduli space. The answer depends entirely on the D terms, as the holomorphic properties of the states, dictated by the superpotential of the worldvolume theory, are independent of Kähler moduli. More concretely, near the orbifold, θ-stability depends explicitly on Kähler moduli, through the FI terms. A criterion for stability based on the periods of the Calabi-Yau, and therefore exact in α', was presented in [16]. Note that, as in geometric engineering, we need of mirror symmetry to provide the exact periods if we want a complete discussion of the lines of marginal stability.

These ideas have a nice mathematical counterpart near the orbifold, known as quiver theory, summarized in the following table

Worldvolume theory	Quiver theory
F-flatness conditions	Quiver with relations
Single boundstate	Schur representation
"Quasi" susy vacuum	θ-stable representation

Let's briefly recall how to obtain the spectrum of boundstates. For more details, see the appendix of [4]. For configurations with $\vec{n} = (n_1, \ldots, n_k)$ fractional branes and where the F-flatness conditions are trivially satisfied (by setting to zero some vev's), the expected dimension of the moduli space is

$$d(\vec{n}) = 1 - \frac{1}{2}\vec{n}^T \cdot C \cdot \vec{n} \tag{3.2}$$

with C the generalized Cartan matrix associated with the quiver. This is quite easy to understand; we count the number of parameters in the gauge group and subtract the number of entries in the matrices representing the vevs of the chiral fields. We call imaginary and real roots those $\vec{n}$ for which $d(\vec{n}) \geq 1$ and $d(\vec{n}) = 0$, respectively. A Schur root is an imaginary or a real root, but the opposite is not true, so after finding all imaginary and real roots, we have to determine which ones of those are Schur roots. We are not aware of a systematic procedure, but for our problem, it will be fairly easy to decide in many cases.

3.1 Range of validity

Before we embark in the study of the spectrum of classical ($g_s = 0$) boundstates of this compactification, we would like to discuss the range of validity of the quiver theory and of θ-stability.

As mentioned in the introduction, a claim central to recent work on BPS D-branes on Calabi-Yau manifolds [16, 4, 6, 1] is that these BPS states can be described by $\mathcal{N} = 1$ theories, even though the constituent parton branes preserve different $\mathcal{N} = 1$ supersymmetries. At an orbifold point typically the central charges are real (the only contribution coming from the B field at the singularity), so they are aligned. As we move away from the orbifold, the central charges of the different fractional branes will be no longer aligned, and eventually it can happen that two fractional branes A and B, that near the orbifold had almost parallel central charges, now have them almost antiparallel, so at that point

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in moduli space the possible boundstate is between A and $\bar{B}$, the antibrane of B. This is discussed in detail in [1]. By the time we reach this point, the quiver theory description has broken down.

On the other hand, θ-stability is only valid at linear order in the FI terms, so it breaks down as soon as the periods are no longer linear in the FI terms.

For the present case, we will show that the field theory particles correspond to BPS states in the string theory that can be described with quiver theory, but θ-stability only gives the lines of marginal stability somewhere near orbifold, and a complete stability analysis would involve the full Π-stability condition.

3.2 The boundstates of the worldvolume theory

The truncated quiver with $2N-2$ nodes can be pictured more conveniently as

$$\begin{tikzcd}
V_1 \arrow[r, shift left] \arrow[r, shift right] & V_2 \\
V_3 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, phantom, "\leftleftarrows"] & V_4 \arrow[u] \\
V_5 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, phantom, "\leftleftarrows"] & V_6 \arrow[u]
\end{tikzcd} \tag{3.3}$$

where the V_i are vector spaces for representations of the quiver. As we will see, it turns out that all the states of the SU(N) SYM theory, can be identified with representations of the previous quiver with the diagonal arrows set to zero,

$$\begin{tikzcd}
V_1 \arrow[r, shift left] \arrow[r, shift right] & V_2 \\
V_3 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & V_4 \arrow[u] \\
V_5 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & V_6 \arrow[u]
\end{tikzcd} \tag{3.4}$$

There are definitely more boundstates in the former quiver, meaning that there are BPS states in the string compactification that don't appear in the SU(N) spectrum. We will see that it is possible to choose the FI terms in such a way that there is a region near the orbifold where these extra states are not present.

First, we are going to study the spectrum of boundstates when only the vertical arrows have non-zero vev,

$$V_{2k-1} \longrightarrow V_{2k-3} \longrightarrow \dots \longrightarrow V_1 \tag{3.5}$$

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The F-flatness conditions are trivially satisfied in this case. The expected dimension of the moduli space of the gauge theory is

$$d = 1 - \left(n_1^2 + n_3^2 + \cdots + n_{2k-1}^2 - n_1 n_3 - \cdots - n_{2k-3} n_{2k-1}\right)$$

So the imaginary roots should satisfy

$$n_1^2 + (n_1 - n_3)^2 + \cdots + (n_{2k-3} - n_{2k-1})^2 + n_{2k-1}^2 \leq 0$$

and the real roots

$$n_1^2 + (n_1 - n_3)^2 + \cdots + (n_{2k-3} - n_{2k-1})^2 + n_{2k-1}^2 = 2$$

We immediately see that there are no non-trivial imaginary roots. For the real roots, the only possibility is that two summands are 1 and the rest 0. A moment's thought shows that all the solutions consist of a chain of adjacent $n_i = 1$ and the rest of the n_j's set to zero. For instance, if we have $n_1 = n_3 = n_5 = 1$, it describes a boundstate with one fractional brane of the first kind, one of the third and one of the fifth. Furthermore, these representations break the gauge group to the diagonal U(1): we start with a gauge group $\mathrm{U}(1) \times \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$, and each nonzero vev breaks the two U(1)'s it transforms under to their diagonal U(1). Therefore, they correspond to boundstates. In the next subsection, we will identify these boundstates with the potentially massless dyons.

Next we consider configurations with only the vevs of a pair of horizontal arrows not zero,

$$V_1 \rightrightarrows V_2 \qquad (3.6)$$

This is known in the math literature as the Kronecker quiver. The superpotential plays no role and the spectrum of boundstates is actually well known [35] (see also the appendix of [4]). The expected dimension of the moduli space is

$$d = 1 - (n_1^2 + n_2^2 - 2n_1 n_2) = 1 - (n_1 - n_2)^2$$

so the imaginary Schur roots must satisfy

$$(n_1 - n_2)^2 \leq 0 \Rightarrow n_1 = n_2$$

and actually only $(1,1)$ is a Schur root [36]. In the next section we will identify it as a W^+ boson with electric charge a simple root of $su(N)$. All the real roots of this quiver are Schur [35] , and they are given by

$$(n_1 - n_2)^2 = 1 \Rightarrow n_1 = n_2 \pm 1$$

Later we will identify these states as the familiar towers of dyons with fixed magnetic charge, when the magnetic charge is a simple root of the algebra.

Now we consider quiver representations with both the horizontal and the vertical representations turned on. The F-flatness conditions are no longer satisfied automatically, so we are in the realm of quivers with relations, and the methods we have been using no

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longer apply. Nevertheless, we will be able to display representations that satisfy F-flatness and break the gauge group to U(1), corresponding to the expected positive charged gauge fields and tower of dyons. We believe that those are the only Schur representations of this quiver compatible with the superpotential, but we don't have a proof of this claim. Consider for concreteness states with magnetic charge given by $\alpha_1 + \alpha_2$. They correspond to representations of

$$\begin{tikzcd} \mathbb{C}^{n_1} \arrow[r, shift left, "X_1"] \arrow[r, shift right, "Y_1"'] & \mathbb{C}^{n_2} \\ \mathbb{C}^{n_1} \arrow[u, "Z_1"'] \arrow[r, shift left, "X_2"] \arrow[r, shift right, "Y_2"'] & \mathbb{C}^{n_2} \arrow[u, "Z_2"] \end{tikzcd} \tag{3.7}$$

the F-flatness conditions (2.1) reduce to

$$Z_1 X_1 = X_2 Z_2 \qquad\qquad Z_1 Y_1 = Y_2 Z_2$$

If we take $n_1 = n_2 = 1$, and nonzero vevs, $Z_1 = Z_2, X_1 = X_2, Y_1 = Y_2$ we satisfy the F-flatness conditions and break the gauge group to the diagonal U(1). These kind of representations correspond to W^+ bosons whose electric charge is a positive, but not simple, root of the algebra. We can describe another solution: take $n_1 = n, n_2 = n+1$ and $X_1 = X_2$ and $Y_1 = Y_2$ to be the Schur representation of the Kronecker quiver for $(n, n+1)$, $Z_1 = \mathbb{I}_n$ and $Z_2 = \mathbb{I}_{n+1}$. It is clear that the F-flatness conditions are satisfied, and it also easy to see that this choice of vevs breaks the gauge group to the diagonal U(1): originally we had $\mathrm{U}(n) \times \mathrm{U}(n+1) \times \mathrm{U}(n) \times \mathrm{U}(n+1)$. By construction X_1, Y_1 break the first $\mathrm{U}(n) \times \mathrm{U}(n+1)$ to its diagonal U(1), and X_2, Y_2 do the same for the second $\mathrm{U}(n) \times \mathrm{U}(n+1)$. Finally, both Z_1 or Z_2 break the $\mathrm{U}(1) \times \mathrm{U}(1)$ we had so far to the diagonal U(1). Therefore we have a boundstate. We will identify them in the field theory with dyons whose magnetic charge is a positive but not simple root of the algebra. Note that the presence of the F-flatness conditions is crucial to avoid the presence of many unwanted states: we could have considered a $(n, n+1, m, m+1)$ representation, with the Schur representations of the Kronecker quiver for the $(n, n+1)$ and $(m, m+1)$ and nonzero matrices Z_1, Z_2. This would break the gauge group to U(1), but in general it does not satisfy the F-flatness conditions.

Finally, we can consider turning on the diagonal arrows of (3.3). As mentioned, states with non zero vevs of these fields don't appear in the SU(N) spectrum, but we don't have a *a priori* reason to discard them from our study. It is immediate that there are new states. For instance, we can turn just a pair of diagonal arrows in (3.3),

$$V_2 \rightrightarrows V_3$$

and this is just a Kronecker quiver (3.6) , which has infinite boundstates. More than that, turning on now just horizontal and diagonal vevs

$$V_1 \rightrightarrows V_2 \rightrightarrows \ldots \rightrightarrows V_k \tag{3.8}$$

there is always a solution satisfying the superpotential constraints, given by setting all the $n_i = 1$. We believe it is the only solution, but we don't have a proof ot this claim.

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3.3 Subrepresentations and domains of stability

In the previous subsection, we have described the possible boundstates of branes we have near the orbifold. To decide where in moduli space each of them is present, we have to check where are they θ-stable. We present the computation for a number of examples. Some of the novel notions of homological algebra that enter the generic picture of [16, 1] are quite easy to understand in this limit.

We will look for subrepresentations and study the domains of stability. Let's start with the boundstates with only vertical arrows. As we just argued, all the non trivial vector spaces at the nodes of the representations have dimension 1, so we will represent them by $\mathbb{C}$. For the sake of concreteness, let's consider a boundstate of 3 fractional branes. Our considerations generalize trivially. There are in this case two subrepresentations

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\simeq"] & \mathbb{C} \arrow[r, "\simeq"] & \mathbb{C} \\ 0 \arrow[r, "0"] \arrow[u, dashed, "0"'] & 0 \arrow[r, "0"] \arrow[u, dashed, "0"'] & \mathbb{C} \arrow[u, dashed, "\simeq"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\simeq"] & \mathbb{C} \arrow[r, "\simeq"] & \mathbb{C} \\ 0 \arrow[r, "0"] \arrow[u, dashed, "0"'] & \mathbb{C} \arrow[r, "\simeq"] \arrow[u, dashed, "\simeq"'] & \mathbb{C} \arrow[u, dashed, "\simeq"'] \end{tikzcd}$$

The notation of these diagrams was introduced in [17]. The top row is the original representation, and the bottom one is the subrepresentation. When the vev of a chiral field is non-zero, we perform a complex gauge transformation to set it to the identity map, denoted by $\simeq$. Vevs not turned on are represented by the 0 map. By definition, there must be an injective map from the subrepresentation to the original representation, denoted here by the dashed vertical arrows. The commutativity of these diagrams is evident. In general, for a boundstate given by a chain of k vector spaces $\mathbb{C}$ and $k-1$ identity maps among them, there will be $k-1$ subrepresentations, being embedded in the original representation "by its end", to ensure the commutativity of the diagram.

We can study now the domain of stability of these representations. Again for concreteness, we focus in the particular example with 3 nodes. We introduce a vector $(\theta_1, \theta_2, \theta_3)$, which must satisfy $n \cdot \theta = 0$, which in this case reduces to $\theta_1 + \theta_2 + \theta_3 = 0$. Stability against decay triggered by the first subrepresentation requires $\theta_3 > 0$, while the second one requires $\theta_2 + \theta_3 > 0$ or equivalently, $\theta_1 < 0$. In the θ-plane we have then two lines of marginal stability for this boundstate.

Let's move now to the boundstates of the Kronecker quiver (3.6). First the imaginary

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root (the W^+ boson) has a single subobject

$$\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "\simeq"] \arrow[r, shift right, "\simeq"'] & \mathbb{C} \\
0 \arrow[u, dashed, "0"'] \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] & \mathbb{C} \arrow[u, dashed, "\simeq"']
\end{tikzcd}$$

This representation is then stable for $\theta_2 > 0$. Next we should consider the subrepresentations of the real roots of this quiver. Displaying them would require some work, as now we are dealing with vectorspaces of arbitrary dimensions.

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, shift left] \arrow[r, shift right] & \mathbb{C}^{n\pm1}
\end{tikzcd}$$

Fortunately, if we just want to know the lines of marginal stability, we don't need that much. For quivers without relations, there is a theorem characterizing Schur roots, due to Schofield [37], which will be quite useful for us. The generic theorem is explained in the appendix of [4] , and in the present case it boils down to saying that (n_1, n_2) is a Schur root iff all its subrepresentations (n'_1, n'_2) satisfy $n'_1/n'_2 < n_1/n_2$. Now consider a particular Schur root (n_1, n_2), and introduce a vector (θ_1, θ_2) such that $n_1\theta_1 + n_2\theta_2 = 0$. Then (n_1, n_2) is θ-stable if $n'_1\theta_1 + n'_2\theta_2 > 0$, i.e., if $\theta_2 > 0$.

Finally, we will consider an example of the domain of stability for boundstates of (3.4), when both horizontal and the vertical arrows are nonzero. For states $(n, n+1, n, n+1)$ we can have subrepresentations (m_1, m_2, m_1, m_2) such that (m_1, m_2) is a subroot of $(n, n+1)$; in particular $m_1/m_2 < n/n+1$

$$\begin{tikzcd}
& \mathbb{C}^n \arrow[rr, shift left] \arrow[rr, shift right] & & \mathbb{C}^{n+1} \\
\mathbb{C}^n \arrow[ur] \arrow[rr, shift left] \arrow[rr, shift right] & & \mathbb{C}^{n+1} \arrow[ur] & \\
& \mathbb{C}^{m_1} \arrow[uu, dashed] \arrow[rr, shift left] \arrow[rr, shift right] & & \mathbb{C}^{m_2} \arrow[uu, dashed] \\
\mathbb{C}^{m_1} \arrow[uu, dashed] \arrow[ur] \arrow[rr, shift left] \arrow[rr, shift right] & & \mathbb{C}^{m_2} \arrow[uu, dashed] \arrow[ur] &
\end{tikzcd}$$

We choose a vector $\vec{\theta}_i$ such that $n(\theta_1+\theta_3)+(n+1)(\theta_2+\theta_4) = 0$. This representation is θ-stable against $(m, m+1, m, m+1)$ if $m_1(\theta_1+\theta_3)+m_2(\theta_2+\theta_4) > 0$. Using $m_1/m_2 < n/n+1$,

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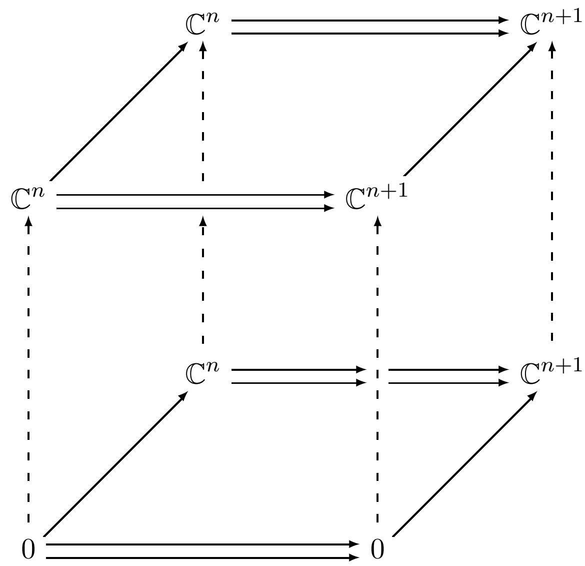
we see that we must require $\theta_2 + \theta_4 > 0$. There is another possible subrepresentation that can trigger a decay,



Note that we can't place the 0's in the other two nodes, for the diagram would not commute. Physically this means that among the decay products, one of them (but not the rest) are triggering the decay [16]. This subrepresentation imposes $n\theta_1 + (n+1)\theta_2 > 0$ for stability of the original representation.

4. Comparison with the $\mathcal{N} = 2$ SU(N) SYM spectrum

In the previous section, we have derived the boundstates of fractional branes, and we studied the jumps in the spectrum near the orbifold, by resorting to a simple linear analysis. We would like now to compare with the results for field theory. This involves a number of issues.

i) As already mentioned, for SU(N) we consider the truncated quiver with $2(N-1)$ nodes. The antiparticles are to be thought of as representations with all the n's negative.

ii) We are going to show that a particular choice of FI terms, reproduces the strong coupling spectrum: only the potentially vanishing states. Note that there are $N(N-1)$ of those, not just the $2(N-1)$ corresponding to the basis of vanishing cycles. More than that, varying the FI terms we will find new states that also have a counterpart in the field theory. A better understanding of why this particular neighborhood of the orbifold reproduces the expected spectrum of the field theory would require a full analysis of the moduli space and periods of this string compactification.

4.1 SU(2)

In this case, we have two possible partons in the theory, and according to the identification we proposed in section 2, they correspond to the monopole $[1,0]$ and the fundamental dyon $[-1,1]$ that go massless in the Seiberg-Witten solution [14]. This amounts to assign to

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the boundstate $(n_1, n_2) = n_1[1,0] + n_2[-1,1] = [n_1 - n_2, n_2]$ magnetic and electric charges given by[8]

$$q_m = n_1 - n_2 \qquad\qquad q_e = n_2$$

The worldvolume theory corresponds to the Kronecker quiver and the spectrum is the following: the imaginary root $(1,1)$; its charges are $[0,1]$, so we identify it with the W^+ boson. Notice that the mathematical statement that there are no (k,k) boundstates, even though they satisfy the dimension formula, corresponds to the statement that there are no particles with charge $[0,k]$ in SU(2) SYM. Next we have the real roots $(n_1, n_1 \pm 1)$. Their charges are $[\pm 1, n_1]$, and we recognize them as the tower of dyons with one unit of magnetic charge.

What can we say about lines of marginal stability in this case? The physical moduli space, in the linear approximation applied in this paper, has as coordinates the two FI terms of the worldvolume gauge theory, ζ_1, ζ_2. For each (n_1, n_2) we introduce a vector (θ_1, θ_2). In the previous section, we have performed the θ-stability analysis for these boundstates. The result was that all of them decay when $\theta_2 > 0$. Now the relation (3.1) between the physical FI terms ζ_i and the θ_i reads in this case

$$\theta_1 = \frac{n_2}{n_1 + n_2}(\zeta_1 - \zeta_2) \qquad\qquad \theta_2 = \frac{n_1}{n_1 + n_2}(\zeta_2 - \zeta_1)$$

So the condition $\theta_2 > 0$ for the different boundstates $(n, n \pm 1)$ translates into a common condition in term of the FI terms, $\zeta_2 > \zeta_1$, even though the map between θ's and ζ's changes for different boundstates. In other words, the linear analysis predicts that all these states decay at the same line of marginal stability. This is precisely what happens for the tower of dyons of SU(2)! [38]. This result has also been derived within the framework of geometric engineering [10].

4.2 SU(3)

Already for SU(3), we are not aware of a detailed description of all the lines of marginal stability. A qualitative new feature is the presence in the moduli space of points, Argyres-Douglas points [39], where mutually non-local particles go massless. The spectrum of BPS states near these points was studied in [40].

The truncated quiver has 4 nodes,

$$\begin{tikzcd} V_1 \arrow[r, shift left] \arrow[r, shift right] & V_2 \\ V_3 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, phantom, "\leftleftarrows"] & V_4 \arrow[u] \end{tikzcd} \tag{4.1}$$

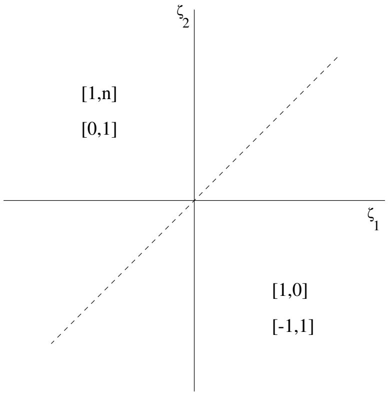

Figure 4: Line of marginal stability for the N=2 orbifold.

Let's start listing the possible states: first we have the four fractional branes $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. They are identified with the basis of vanishing cycles.

[8]Our notation is as follows, (n_1, n_2) represents a boundstate of n_1 monopoles and n_2 fundamental dyons; $[q_m, q_e]$ represents a state of magnetic and electric charges q_m and q_e.

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The remaining two potentially massless dyons come from states with just the vertical arrows turned on: $(1,0,1,0)$ and $(0,1,0,1)$. This is the set of $N(N-1) = 3 \cdot 2 = 6$ potentially massless dyons.

The positively charged gauge bosons are also easily identified: they correspond to bound states with the horizontal arrows turned on:$(1,1,0,0)$ and $(0,0,1,1)$ are the ones with electric charge α_1 and α_2, and $(1,1,1,1)$ is the one with electric charge $\alpha_1+\alpha_2$.

The towers of dyons with magnetic charges α_1 and α_2 are bound states of the Kronecker quivers: $(n, n\pm 1, 0, 0)$ and $(0,0,n,n\pm 1)$. For the positive non-simple root $\alpha_1+\alpha_2$ we have states described by (3.7) , $(n, n\pm 1, n, n\pm 1)$. Finally, we have states that are not expected in the field theory: $(1,1,1,0)$, $(0,1,1,1)$, $(1,1,1,1)$, $(0,n,n\pm 1,0)$.

Let's describe now the lines of marginal stability for the different states. Using the results we obtained in the previous section, we see that the two potentially massless dyons $(1,0,1,0)$ and $(0,1,0,1)$ are present in the spectrum as long as $\zeta_1 > \zeta_3$ and $\zeta_2 > \zeta_4$, respectively. The W^+ bosons with electric charges α_1 and α_2 exist as long as $\zeta_2 > \zeta_1$ and $\zeta_4 > \zeta_3$, respectively. The W^+ boson with electric charge $\alpha_1+\alpha_2$ requires $\zeta_1+\zeta_2 > \zeta_3+\zeta_4$ and $\zeta_2+\zeta_4 > \zeta_1+\zeta_3$.

The towers of dyons with magnetic charge α_1 and α_2 are stable as long as $\zeta_2 > \zeta_1$ and $\zeta_4 > \zeta_3$. The dyons with magnetic charge $\alpha_1+\alpha_2$ require $\zeta_2+\zeta_4 > \zeta_1+\zeta_3$ and $n(\zeta_1-\zeta_3)+(n\pm 1)(\zeta_2-\zeta_4) > 0$.

Finally the states that are not present in field theory, $(1,1,1,0)$, $(0,n,n\pm 1,0)$ and $(0,1,1,1)$ will appear when $3\zeta_3 > \zeta_1+\zeta_2+\zeta_3 > 3\zeta_1$, $\zeta_3 > \zeta_2$ and $3\zeta_4 > \zeta_2+\zeta_3+\zeta_4 > 3\zeta_2$, respectively.

We see then that if we consider the region with $\zeta_1 > \zeta_3$, $\zeta_2 > \zeta_3$ and $\zeta_2 > \zeta_4$, we always have the potentially massless dyons present in the spectrum and none of the states that don't appear in the field theory.

4.3 SU(N)

For generic SU(N), we have first the boundstates consisting of only "back two" chiral fields (vertical arrows in (3.4)). Let's count how many of them we have. First, we have the 2(N-1) nodes. For each node except the last two, we have a boundstate of just two fractional branes, i.e. involving a single arrow; there are 2(N-2) of those. Furthermore, there are 2(N-4) boundstates of 3 fractional branes, involving two arrows, and so on. All in all, there are N(N-1) such boundstates. From the identification of the 2(N-1) fractional branes with the basis of vanishing cycles, it follows that these N(N-1) boundstates are the spectrum of potentially massless dyons. According to [19], this is the strong coupling spectrum of SU(N) SYM. The correspondence of our boundstates with the rational A-type boundary states of the Gepner model of [19] is quite clear: the nodes of the quiver are the $L=0$ boundary states. The boundstates with a single arrow correspond to the $L=1$ boundary states, the ones with two arrows are the $L=2$ boundary states, and so on.

A very similar counting, but now of representations with horizontal arrows turned on, yields $N(N-1)/2$ postively charged gauge bosons. Note that this analysis can't recover the neutral gauge bosons of the field theory as they don't arise from branes wrapping cycles.

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On top of these states, we also have as potential states in the spectrum all the boundstates with the "forward one" arrows. When there are only two nodes, they give the tower of dyons for the different simple roots of the algebra. When we have more than one pair of horizontal arrows, we obtain the tower of dyons for positive non-simple root. The analysis of the domains of stability of the different states, could be carried out as for SU(3). In particular, if we take our FI terms satisfying $\zeta_i > \zeta_{i+1}$, the only states present in that region are the potentially massless dyons, so in this negihboorhod of the orbifold the spectrum coincides with the expected BPS spectrum of the field theory.

5. Conclusions

In this paper we have related some of the ideas that have been recently brought up in the study of BPS branes on Calabi-Yau varieties to the more familiar setting of $\mathcal{N} = 2$ field theories. To do so we started with the Calabi-Yau geometry used to geometrically engineer pure SU(N) SYM, and considered the non-geometric phase, an orbifold. The advantage of studying this phase is that it is then very easy to obtain a set of branes that constitute a basis for the K-theory of the Calabi-Yau, namely the fractional branes at the orbifold. These fractional branes are identified with dyons of the field theory whose magnetic charge is a simple root of the algebra. The whole spectrum can be thought of as boundstates of a finite number of these states. We have displayed these boundstates, and performed a study of their domains of existence near the orbifold.

As already mentioned, a crucial step in our derivation of the orbifold point was that the toric diagram was simplicial. As this is not the case for general $\mathcal{N} = 2$ theories with matter, it is not straightforward to generalize the kind of analysis we have performed here. The blowdown limit can still be derived using the methods of [26], but it won't be an orbifold. On the mirror side, there is a proposed Gepner model for SU(N_c) with $N_f = N_c - 1$ flavors [41], so one expects this case to present some simplification on the type IIA side also.

Finally, on a more general level, one can ask what are the structures behind the spectrum of BPS states, both in string theory compactifications and in $\mathcal{N} = 2$ field theories. On one hand, we can consider the derived category [1], which is manifestly independent of vector moduli space. Another possibility is to consider the algebra of BPS states: a universal property of the spectrum of BPS states for any theory is that they form an algebra [42], which depends on vector moduli space. In [42], the definition for the product of that algebra was given in terms of a scattering process, and obvious phase space considerations force an analytic continuation to complex momenta. In [17] , a slightly different interpretation of the algebra of BPS states was presented: if the coefficient c_{ij}^k in the algebra is not zero, we say that ϕ_i and ϕ_j can form a boundstate ϕ_k, with ϕ_i being a subobject of ϕ_k. Notice that the definition is not symmetric in ij.

In [17] the notion of algebra of BPS states was reformulated near orbifold points. The results presented here could be then used to study the algebra of BPS states for field theories. Take for instance $\mathcal{N} = 2$ SU(2) SYM. Its spectrum is given by the Kronecker quiver of (3.6), whose Schur roots correspond to the stable sheaves of $\mathbb{P}^1$. Indeed, if we

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identify the rank of a sheaf in $\mathbb{P}^1$ with magnetic charge and the first Chern number with the electric charge, we see that the tower of dyons correspond to line bundles $\mathcal{O}(k)$ on $\mathbb{P}^1$, and the W^+ corresponds to the skyscraper sheaf of length one, $\mathcal{O}_{\mathbb{P}}$. We have the exact sequence,

$$0 \to \mathcal{O}(n) \to \mathcal{O}(n+1) \to \mathcal{O}_{\mathbb{P}} \to 0$$

This can be read as saying that a $[1,n]$ dyon and a W^+ boson can form a $[1,n+1]$ dyon, with the $[1,n]$ dyon being a subobject, but not the W^+ boson. Notice that we can not have the reverse exact sequence, as we can't have an injective map from torsion sheaves to torsion free sheaves. This means [42, 17] that the structure constants $c_{n,W}^{n+1} \neq 0$ and $c_{W,n}^{n+1} = 0$. It would be very interesting to determine the algebra beyond this homologic approximation, and elucidate its dependence on the coupling constant.

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